EIC Kinematics for Fixed Target Experimentalists

E. Long (UNH)

Abstract
This note demonstrates how to relate kinematic variables used in fixed-target electron beam experiments to those used in an electron-ion collider. It also demonstrates how nucleon-scaling of these variables commonly used in Jefferson Lab experiments can be translated to collider definitions.
1 Laboratory Frame Definitions

Throughout this tech note we’ll be using natural units where $\hbar = c = 1$ and the laboratory frame unless otherwise specified. At fixed-target facilities, such as Jefferson Lab, the majority of the momentum is carried by the electron and so variables are defined according to the electron beam where $+\hat{z}$ is oriented along the electron beam. The main kinematic variables used are the electron polar scattering angle $\theta_e$, the electron azimuthal scattering angle $\phi_e$, the initial electron 4-momenta $k = (E_e, k)$, the final electron 4-momenta $k' = (E'_e, k')$, the initial target 4-momenta $p = (E_h, p)$ and the final hadronic 4-momenta $p' = (E'_h, p')$. Given that the target is fixed, $E_h = m_h$ where $m_h$ is the target mass and $p = 0$ (unless hitting quarks or a nucleon inside of an $N > 1$ nucleus, in which case it is on the scale of the Fermi momentum and still significantly less than the nuclear mass). The fixed-target 4-momenta are described by

$$k_{\text{JLab}} = \begin{pmatrix} E_e \\ 0 \\ 0 \\ E_e \end{pmatrix}, \quad k'_{\text{JLab}} = \begin{pmatrix} E'_e \\ E'_e \sin \theta_e \cos \phi_e \\ E'_e \sin \theta_e \sin \phi_e \\ E'_e \cos \theta_e \end{pmatrix}$$

(1)

$$p_{\text{JLab}} = \begin{pmatrix} E_h \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p'_{\text{JLab}} = \begin{pmatrix} E'_h \\ p'_{hx} \\ p'_{hy} \\ p'_{hz} \end{pmatrix}$$

(2)

For electron-ion collider experiments, the majority of the momentum is carried by the ion beam and so variables are defined such that $+\hat{z}$ is oriented along the ion beam and against the electron beam. A detailed description is given in [1]. Given that the scattered electron is then often knocked back towards the direction it came from, the scattering angle $\theta_1$ is used. We could also use the same scattering angle defined for fixed-target experiments, $\theta_e$, where $\theta_1 + \theta_e = \pi$. It also makes sense to define $\gamma$ as the scattering angle of the struck quark or hadron. $H'$ and $p'$ are summed over all particles $h$ that are produced in the final

Figure 1: Common kinematic variables for Jefferson Lab experiments.
Figure 2: Common kinematic variables for electron-ion collider experiments.

state. The EIC 4-momenta are thus described by

$$k_{\text{EIC}} = \begin{pmatrix} E_e \\ 0 \\ 0 \\ -E_e \end{pmatrix}, \quad k'_{\text{EIC}} = \begin{pmatrix} E'_e \\ E'_e \sin \theta_e \cos \phi_e \\ E'_e \sin \theta_e \sin \phi_e \\ E'_e \cos \theta_e \end{pmatrix} = \begin{pmatrix} E'_e \\ E'_e \sin \theta_e \cos \phi_e \\ E'_e \sin \theta_e \sin \phi_e \\ -E'_e \cos \theta_e \end{pmatrix}$$ (3)

$$p_{\text{EIC}} = \begin{pmatrix} E_h \\ 0 \\ 0 \\ E_h \end{pmatrix}, \quad p'_{\text{EIC}} = \begin{pmatrix} E'_h \\ p'_{hx} \\ p'_{hy} \\ p'_{hz} \end{pmatrix}$$ (4)

Note that the two major differences between the JLab 4-momenta and the EIC 4-momenta are the sign reversal of $k_z$ that comes from the change of $\hat{z}$ reference from the electron beam to the ion beam, and the non-zero $p_z$ in the case of an ion beam.

For collider experiments, it’s often easier to do measurements and thus determine kinematics from the hadronic flow, which is discussed in detail in [1]. For this technical note, we’ll only concern ourselves with the lepton flow in order to connect definitions through the Lorentz-invariant quantities where the final result is insensitive to which method is used.

If we define an EIC experiment using the fixed-target definitions for $+\hat{z}$ oriented along the electron beam and use the electron scattering angle $\theta_e$ while also trading a fixed target for an ion beam oriented along $-\hat{z}$, we get the following definitions of 4-momenta that will be used throughout the rest of this technical note.

$$k = \begin{pmatrix} E_e \\ 0 \\ 0 \\ E_e \end{pmatrix}, \quad k' = \begin{pmatrix} E'_e \\ E'_e \sin \theta_e \cos \phi_e \\ E'_e \sin \theta_e \sin \phi_e \\ E'_e \cos \theta_e \end{pmatrix}$$ (5)
\[ p = \begin{pmatrix} E_h \\ 0 \\ 0 \\ -E_h \end{pmatrix} \quad \quad p' = \begin{pmatrix} E'_h \\ p'_{hx} \\ p'_{hy} \\ -p'_{hz} \end{pmatrix} \quad (6) \]

2 Mandelstam Variables

In order to calculate the Lorentz invariant quantities in each case, we’ll need to use the Mandelstam variables that are defined by

\[ t = (k' - k)^2 = (p' - p)^2 \quad (7) \]
\[ s = (k + p)^2 = (k' + p')^2 \quad (8) \]
\[ u = (k' - p)^2 = (p' - k)^2 \quad (9) \]

In fixed-target experiments, we are most familiar with \( Q^2 = -t \). For collider experiments, the center-of-mass energy \( \sqrt{s_{\text{EIC}}} = \sqrt{4E_eE_h} \) is often used as a frame of reference.

In fixed target experiments, \( E_h = m_h \) (target mass), \( p^2 \neq 0 \), and \( \sqrt{s_{\text{JLab}}} = \sqrt{2m_hE_e + m_h^2} \).

3 Lorentz Invariant Terms

We define the Lorentz invariant terms either as functions of the Mandelstam variables or of the individual 4-momenta where we also define the \( q \)-vector

\[ q = k - k' = d' - d = \begin{pmatrix} E_e - E'_e \\ -E'_e \sin \theta_e \cos \phi_e \\ -E'_e \sin \theta_e \sin \phi_e \\ E_e - E'_e \cos \theta_e \end{pmatrix} \quad (10) \]

This gives us our fundamental definitions

\[ Q^2 = -t = -q^2 \quad (11) \]
\[ x = \frac{Q^2}{2pq} \quad (12) \]
\[ y = \frac{pq}{pk} \quad (13) \]
\[ W^2 = (p')^2 = (p + q)^2 \quad (14) \]

Note that in the case of colliders, we can directly relate \( x, y, \) and \( W^2 \) to the Mandelstam variables by

\[ x = \frac{-t}{u + s} \quad (15) \]
\[ y = \frac{u + s}{s} = \frac{Q^2}{sx} \] (16)

\[ W^2 = s + t + u \] (17)

but this only works since \( p^2 = 0 \), which is not the case in fixed-target experiments and must be taken into account.

For \( Q^2 \), both \( k^2 \) and \( k'^2 \) are zero leaving \( 2kk' \) as the only non-zero term, which leads to the familiar definition of

\[ Q^2 = 4E_eE'_e \sin^2 \left( \frac{\theta_e}{2} \right) \] (18)

However, note that if \( \theta_1 \) is used it takes the form \( Q^2 = 4E_eE'_e \cos^2 \left( \frac{\theta_1}{2} \right) \), which is equivalent to Eq. (18).

The familiar definitions of Bjorken-\( x \) are significantly different between EIC and fixed-target experiments and deserve particular attention. From Eq. (12), the denominator \( 2pq \) gains an extra term in EIC experiments that is absent from fixed-target experiments due to the non-zero component of \( p_z \). This term is similar to the energy transfer \( \nu = E_e - E'_e \), except that it only accounts for the energy transferred along the \( \hat{z} \) direction and will be defined as

\[ \nu_z = E_e - E'_e \cos \theta_e \] (19)

which leads to a final definition of \( x \) as

\[ x = \frac{Q^2}{2E_h(\nu + \nu_z)} \] (20)

where the only allowed values are \( 0 \leq x \leq 1 \).

As mentioned above, for fixed-target experiments \( p_z = 0 \) leads to \( \nu_z = 0 \) and \( E_h = m_h \), which causes Eq. (20) to become the more familiar

\[ x_{\text{Fixed}} = \frac{Q^2}{2m_h\nu} \] (21)

It is also important to note that often times in Jefferson Lab experiments that \( x_{\text{Fixed}} \) is scaled to the nucleon mass \( (m_p) \)

\[ x_{\text{JLab}} = \frac{Q^2}{2m_p\nu} = N x_{\text{Fixed}} \] (22)

such that the quasi-elastic peak is always centered at \( x_{\text{JLab}} = 1 \) and the nuclear elastic peak is centered around \( x_{\text{JLab}} = N \) where \( N \) is the number of nucleons in the target. It is non-trivial to scale the collider \( x \) to the nucleon mass (since \( p \neq 0 \) and \( E_h = \sqrt{m^2 + p^2} \)), but we can easily scale it to the average nucleon energy in the ion beam where \( E_h = NE_p \), which gives

\[ x = \frac{Q^2}{2NE_p(\nu + \nu_z)} = \frac{x_p}{N} \] (23)
where

\[ x_p = \frac{Q^2}{2E_p(\nu + \nu_z)} \]  

(24)

has allowed values from \( 0 \leq x_p \leq N \). This also becomes identical to Eq. (22) in fixed-target experiments where \( p_z = 0 \) causes \( \nu_z = 0 \) and \( E_p = m_p \).

Similarly, there are extra terms in \( y \) when doing a collider experiment that are non-existent when doing a fixed-target experiment. Following Eq. (13), these differences again rise from \( p_z \neq 0 \) in a collider experiment in both the \( pq \) and \( pk \) terms. For JLab experiments, the common definitions are

\[ p_{JLab}^2 = E_h \nu \]

(25)

\[ p_{JLab}^2 = E_{e}E_h \]

(26)

\[ y_{JLab} = \frac{\nu}{E_{e}} = 1 - \frac{E'_e}{E_e} \cos^2 \left( \frac{\theta_e}{2} \right) \] 

(27)

When we replace the fixed target with an ion beam, these change to

\[ pq = E_h(\nu + \nu_z) \]

(28)

\[ pk = 2E_eE_h \]

(29)

\[ y = \frac{\nu + \nu_z}{2E_{e}} = 1 - \left( \frac{E'_e}{E_e} \right) \left( \frac{1 + \cos \theta_e}{2} \right) = 1 - \frac{E'_e}{E_e} \cos^2 \left( \frac{\theta_e}{2} \right) \] 

(30)

The invariant mass \( W \) is unique in that the fixed-target definitions gain an extra term that cancels in the EIC definitions. For fixed targets, \( p_z = 0 \) causes \( p^2 = E_h^2 = m_h^2 \) whereas in the collider \( p_z = |p| \) causes \( p^2 = 0 \). From Eq. (14) for a fixed-target experiment we get

\[ p_{JLab}^2 = m_h^2 \]

\[ 2p_{JLab}q_{JLab} = 2E_h\nu = 2m_h\nu \]

\[ q_{JLab}^2 = -Q^2 \] 

(31)

\[ W_{JLab}^2 = m_h^2 + 2m_h\nu - Q^2 \] 

(32)

and for a collider experiment we get

\[ p^2 = 0 \]

\[ 2pq = 2E_h(\nu + \nu_z) \]

\[ q^2 = -Q^2 \] 

(33)

\[ W^2 = 2E_h(\nu + \nu_z) - Q^2 \] 

(34)

The collider definition of \( W^2 \) can then be simply related to \( x \) and \( Q^2 \) by

\[ W^2 = Q^2 \frac{1 - x}{x} \] 

(35)

but this definition should be considered cautiously by fixed-target experimentalists, as we would have to modify it as

\[ W_{JLab}^2 - m_p^2 = Q^2 \frac{1 - x_{JLab}}{x_{JLab}} \] 

(36)

if scaling to the nucleon mass, or replace \( m_p \) with the mass of the target \( m_h \) and \( x_{JLab} \) with \( x_{Fixed} \) for the total invariant mass.
References